Countable Fraïssé Categories

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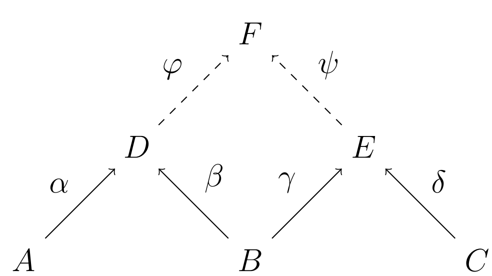
In this note we will discuss the topological groups that naturally arise when considering countable Fraïssé categories. We will also characterize when two categories give rise to isomorphic topological groups.

1 Amalgamation of Categories

A *countable* category is a category A with $\mathrm{Arr}(A)$ countable; in particular, this implies that $\mathrm{Ob}(A)$ is countable. A category A is *directed* if for every $a, b \in \mathrm{Ob}(A)$, there is $c \in \mathrm{Ob}(A)$ with $\mathrm{Hom}(a, c)$ and $\mathrm{Hom}(b, c)$ non-empty. For any two arrows f, g with common domain, we say that (r, s) *amalgamates* (f, g) if $r \circ f = s \circ g$; the category A has *amalgamation* if all pairs of arrows with common domain can be amalgamated. A subset $X \subset \mathrm{Ob}(A)$ is *cofinal* if for each $a \in \mathrm{Ob}(A)$, there is $x \in X$ with $\mathrm{Hom}(a, x)$ non-empty. For any two categories A, B, we write $A \overset{\sim}{\to} B$ if there is a full, faithful functor from A to B which is injective on objects and whose image is cofinal in B. We will often call such a functor a *cofinal embedding*. Observe that the composition of two cofinal embeddings is itself a cofinal embedding, allowing us to give any set of categories a category structure. In what follows, any category whose objects are categories will use cofinal embeddings as the arrows unless explicitly stated otherwise.

Consider then the category $\mathcal{C}$ of countable, directed categories. For $A, B \in \mathrm{Ob}(\mathcal{C})$, write $A \sim B$ if there is $C \in \mathrm{Ob}(\mathcal{C})$ with $\mathrm{Hom}(A, C)$ and $\mathrm{Hom}(B, C)$ non-empty. The relation $\sim$ is certainly reflexive and symmetric. Transitivity is not obvious; let $A, B, C, D, E \in \mathrm{Ob}(\mathcal{C})$ and suppose $\alpha \in \mathrm{Hom}(A, D)$, $\beta \in \mathrm{Hom}(B, D)$, $\gamma \in \mathrm{Hom}(B, E)$, $\delta \in \mathrm{Hom}(C, E)$. If we could find (φ, ψ) amalgamating (β, γ), then letting F be the common image of φ, ψ, we would have $\varphi \circ \beta \in \mathrm{Hom}(A, F)$, $\psi \circ \gamma \in \mathrm{Hom}(C, F)$.



Theorem 1.1. *The category $\mathcal{C}$ has amalgamation. In particular, $\sim$ is an equivalence relation.*

Proof. Suppose φ and ψ are cofinal embeddings of A into B and C, respectively. It will be simpler to identify A with its image in each case, so that we may think of A as a full, cofinal subcategory of both B and C. We will build a category D containing B and C such that the containments are cofinal embeddings which witness amalgamation. We set $\mathrm{Ob}(D) = \mathrm{Ob}(B) \cup \mathrm{Ob}(C)$. As we want the inclusions to be full embeddings, we set $\mathrm{Hom}_D(a, b) = \mathrm{Hom}_B(a, b)$ for any two objects $a, b \in \mathrm{Ob}(B)$, similarly for $a, b \in \mathrm{Ob}(C)$.

It remains to define $\mathrm{Hom}(b, c)$ when $b \in \mathrm{Ob}(B) \setminus \mathrm{Ob}(A)$ and $c \in \mathrm{Ob}(C) \setminus \mathrm{Ob}(A)$, or vice versa. We let $\mathrm{Hom}(b, c)$ consist of all pairs $f_1 \cdot f_0$ with $f_0 \in \mathrm{Hom}(b, a)$ and $f_1 \in \mathrm{Hom}(a, c)$ for any $a \in \mathrm{Ob}(A)$; we set $f_1 \circ f_0 = f_1 \cdot f_0$. If $a_1, a_2 \in \mathrm{Ob}(A)$ and $f \in \mathrm{Hom}(b, a_1)$, $g \in \mathrm{Hom}(a_1, a_2)$, $h \in \mathrm{Hom}(a_2, c)$ we declare that $h \cdot (g \circ f) = (h \circ g) \cdot f$; this ensures that the resulting category is associative. We define composition as follows: If $g \in \mathrm{Hom}(d, b)$ for $d \in \mathrm{Ob}(B)$, we set $(f_1 \cdot f_0) \circ g = f_1 \circ (f_0 \circ g)$ (note that this is defined whether or not $d \in \mathrm{Ob}(A)$). Similarly if $h \in \mathrm{Hom}(c, e)$ for $e \in \mathrm{Ob}(C)$, we set $h \circ (f_1 \cdot f_0) = (h \circ f_1) \circ f_0$. If $\delta \in \mathrm{Ob}(B) \setminus \mathrm{Ob}(A)$ and we have $g_1 \cdot g_0 \in \mathrm{Hom}(c, \delta)$, we set $(g_1 \cdot g_0) \circ (f_1 \cdot f_0) = g_1 \circ (g_0 \circ f_1) \circ f_0$.

We need to verify that composition is well defined; we will verify one instance here, as others are similar to show. Suppose $f_1 \cdot f_0 = g_1 \cdot g_0 \in \mathrm{Hom}(b, c)$ and $h_1 \cdot h_0 \in \mathrm{Hom}(c, d)$, with $b, d \in \mathrm{Ob}(B) \setminus \mathrm{Ob}(A)$ and $c \in \mathrm{Ob}(C) \setminus \mathrm{Ob}(A)$. By repeated application, it is enough to consider the case where $g_0 \in \mathrm{Hom}(b, a_1)$, $p \in \mathrm{Hom}(a_1, a_2)$, $f_1 \in \mathrm{Hom}(a_2, c)$, and $f_0 = p \circ g_0$, $g_1 = f_1 \circ p$. Then we have

$$\begin{aligned} h_1 \circ (h_0 \circ f_1) \circ f_0 &= h_1 \circ [(h_0 \circ f_1) \circ (p \circ g_0)] \\ &= h_1 \circ [((h_0 \circ f_1) \circ p) \circ g_0] \\ &= h_1 \circ [(h_0 \circ (f_1 \circ p)) \circ g_0] \\ &= h_1 \circ (h_0 \circ g_1) \circ g_0. \end{aligned}$$

D is certainly countable. To see that it is directed, say $b \in \mathrm{Ob}(B)$ and $c \in \mathrm{Ob}(C)$. By cofinality of A, pick $a_1, a_2 \in \mathrm{Ob}(A)$ and $f \in \mathrm{Hom}(b, a_1)$, $g \in \mathrm{Hom}(c, a_2)$. As A is directed, we are done. As an added bonus, note that if A, B, C all have amalgamation, then so does D. Without loss of generality pick $b \in \mathrm{Ob}(B)$, and let f_1, f_2 have domain b. By cofinality of A, we can find g_1, g_2 such that $g_i \circ f_i$ has range in A. Now use amalgamation in B. ☐

2 Monicity

A category A is *Fraïssé* if A is directed and has amalgamation. We will see shortly that each countable Fraïssé category A gives rise to a topological group $G(A)$, the automorphism group of any Fraïssé sequence in A. It is our goal to characterize when $G(A) = G(B)$; we will find that up to a certain closure operator on countable Fraïssé categories, this occurs when $A \sim B$. More precisely, we will define the closure $\bar{A}$ of any countable Fraïssé category A such that $G(A) = G(B)$ exactly when $\bar{A} \sim \bar{B}$. The closure operator can be though of as occurring in two steps. Morally speaking, the first step identifies arrows which "ought" to be the same. The second step then introduces arrows which "ought" to be present. We will tackle the first step now.

A category is *monic* if for all arrows, $x \circ f = x \circ g \Rightarrow f = g$. For A a Fraïssé category, define the relation M on arrows where $M(f, g)$ iff there is x with $x \circ f = x \circ g$.

Proposition 2.1. *The relation M is an equivalence relation which respects composition.*

Hence the first step of our closure operation will be forming the category A/M, which will be called the *monic closure*.

Proof. Suppose $M(f, g)$ and $M(g, h)$ as witnessed by arrows x, y respectively. Observe that x and y have common domain; let (r, s) amalgamate (x, y). Then $r \circ x \circ f = r \circ x \circ g = s \circ y \circ g = s \circ y \circ h = r \circ x \circ h$. Hence $M(f, h)$. Now suppose $M(f_0, f_1)$ and $M(g_0, g_1)$ as witnessed by p, q, respectively. Further suppose that $\mathrm{Ran}(f_i) = \mathrm{Dom}(g_i)$. We want to show that $M(g_0 \circ f_0, g_1 \circ f_1)$. Let (r, s) amalgamate $(p, q \circ g_0)$. Then $s \circ q \circ g_0 \circ f_0 = r \circ p \circ f_0 = r \circ p \circ f_1 = s \circ q \circ g_0 \circ f_1 = s \circ q \circ g_1 \circ f_1$. □

Proposition 2.2. *The quotient A/M is a monic Fraïssé category.*

Proof. Given an arrow $f \in \mathrm{Arr}(A)$, let Mf denote its M equivalence class. For $a, b \in \mathrm{Ob}(A) = \mathrm{Ob}(A/M)$, let $f \in \mathrm{Hom}_A(a, c)$ and $g \in \mathrm{Hom}_A(b, c)$. Then $Mf \in \mathrm{Hom}_{A/M}(a, c)$ and $Mg \in \mathrm{Hom}_{A/M}(b, c)$. Thus A/M is directed. Now let a, b, c be objects and $F \in \mathrm{Hom}_{A/M}(a, b)$, $G \in \mathrm{Hom}_{A/M}(b, c)$. Pick representatives $f \in F$, $g \in G$, and let (r, s) amalgamate (f, g) in A. Then (Mr, Ms) amalgamates $(Mf, Mg) = (F, G)$ in A/M. Hence A/M has amalgamation. Lastly, suppose $X \circ F = X \circ G$. Pick representatives x, f, g. Then $M(x \circ f, x \circ g)$, so for some y, we have $y \circ x \circ f = y \circ x \circ g$. Hence $M(f, g)$ and $F = G$, showing that A/M is monic. □

We also want to show $A \sim B$ implies $A/M \sim B/M$. It is enough to show the following.

Proposition 2.3. *Suppose A, B are Fraïssé categories and $A \xrightarrow{\sim} B$. Then $A/M \xrightarrow{\sim} B/M$.*

Proof. Identify A with its image in B. Let $a, b \in \mathrm{Ob}(A)$ and suppose $M_B(a, b)$ as witnessed by $x \in \mathrm{Arr}(B)$. By cofinality of A, find y with $y \circ x \in \mathrm{Arr}(A)$. Then $y \circ x$ witnesses $M_A(a, b)$. □

3 Localization

Let A be a monic Fraïssé category, and suppose $f, g \in \mathrm{Arr}(A)$ have common range. We say that g *stabilizes* f, or that f is *g-stable*, if for any arrows p_1, p_2 with $p_1 \circ g = p_2 \circ g$, we have $p_1 \circ f = p_2 \circ f$. We say that (g, f) is a *stable pair*. The prototypical example of a stable pair is any pair of the form $(g, g \circ x)$. The idea behind forming the *localization* of

A is that these should be the only examples of stable pairs. In particular, if f is g-stable and we cannot write $f = g \circ x$, we want to introduce a map $g^{-1}f$ such that $g \circ g^{-1}f = f$.

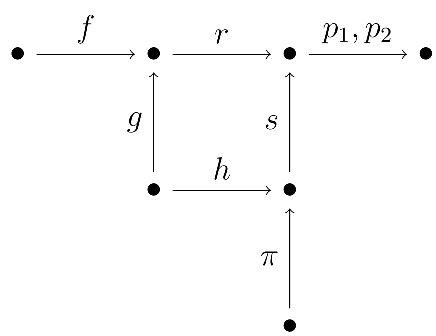
When is it the case that $g_1^{-1}f_1 = g_2^{-1}f_2$? Certainly the g_i must have common domain, as well as the f_i. So let (r, s) amalgamate (g_1, g_2). Then assuming the above equality, we find that $r \circ f_1 = s \circ f_2$, i.e. (r, s) also amalgamates (f_1, f_2). This motivates the following.

Definition 3.1. *We say that* $g_1^{-1}f_1 \doteq g_2^{-1}f_2$ *iff for any* (r, s) *amalgamating* (g_1, g_2)*, we also have that* (r, s) *amalgamates* (f_1, f_2)*.*

Proposition 3.2. $\doteq$ *is an equivalence relation on stable pairs.*

Proof. Certainly $\doteq$ is symmetric. If f is g-stable, then by definition $g^{-1}f \doteq g^{-1}f$. To show transitivity, suppose $g_1^{-1}f_1 \doteq g_2^{-1}f_2$ and $g_2^{-1}f_2 \doteq g_3^{-1}f_3$. Say (r, s) amalgamates (g_1, g_3). Let (x, y) amalgamate $(r \circ g_1, g_2) = (s \circ g_3, g_2)$. It follows that $(x \circ r, y)$ amalgamates (g_1, g_2), and $(x \circ s, y)$ amalgamates (g_3, g_2). By assumption, $(x \circ r, y)$ amalgamates (f_1, f_2) and $(x \circ s, y)$ amalgamates (f_3, f_2). But now we have $x \circ r \circ f_1 = x \circ s \circ f_3$. By monicity, we are done. □

The localization $\bar{A}$ will have $\mathrm{Ob}(\bar{A}) = \mathrm{Ob}(A)$, and $\mathrm{Arr}(\bar{A})$ will consist of all equivalence classes of stable pairs. If $f \in \mathrm{Arr}(A)$ has range $a \in \mathrm{Ob}(A)$, we will often identify f with the pair $(1_a, f) \in \mathrm{Arr}(\bar{A})$. Now for composition; how do we define $\pi^{-1}h \circ g^{-1}f$? Let's try the following: let (r, s) amalgamate (g, h). Since (g, f) and (π, h) are both stable pairs, it follows that $r \circ f$ is $(s \circ \pi)$-stable. Define $\pi^{-1}h \circ g^{-1}f = (s \circ \pi)^{-1}(r \circ f)$.



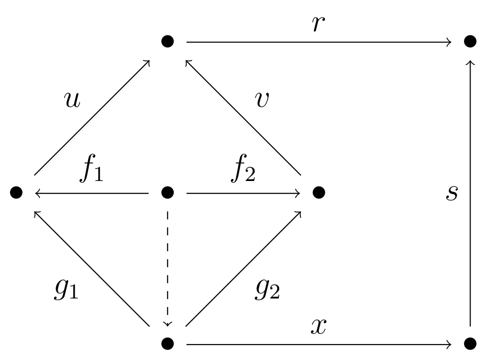
We need to check that this respects $\doteq$ and is independent of the choice of (r, s). So say $g_1^{-1}f_1 \doteq g_2^{-1}f_2$ and $\pi_1^{-1}h_1 \doteq \pi_2^{-1}h_2$; furthermore, suppose (r_1, s_1) amalgamates (g_1, h_1) and (r_2, s_2) amalgamates (g_2, h_2). We need to show that $(s_1 \circ \pi_1)^{-1}(r_1 \circ f_1) \doteq (s_2 \circ \pi_2)^{-1}(r_2 \circ f_2)$. So let (u, v) amalgamate $(s_1 \circ \pi_1, s_2 \circ \pi_2)$. We need to show that (u, v) amalgamates $(r_1 \circ f_1, r_2 \circ f_2)$, i.e. that $u \circ r_1 \circ f_1 = v \circ r_2 \circ f_2$.

$$
\begin{aligned}
& u \circ s_1 \circ \pi_1 = v \circ s_2 \circ \pi_2 \\
\Rightarrow\ & u \circ s_1 \circ h_1 = v \circ s_2 \circ h_2 \quad (\pi_1^{-1}h_1 \doteq \pi_2^{-1}h_2) \\
\Rightarrow\ & u \circ r_1 \circ g_1 = v \circ r_2 \circ g_2 \\
\Rightarrow\ & u \circ r_1 \circ f_1 = v \circ r_2 \circ f_2 \quad (g_1^{-1}f_1 \doteq g_2^{-1}f_2).
\end{aligned}
$$

Draw a large commutative diagram to convince yourself that composition is associative. We say a category A is *stable* if for all $g \in \mathrm{Arr}(A)$, the only g-stable maps are of the form $g \circ x$.

Proposition 3.3. *If A is a monic Fraïssé category, then $\bar{A}$ is a monic, stable Fraïssé category.*

Proof. It is easy to see that $\bar{A}$ is a Fraïssé category. For monicity, it is enough to show that $x \circ g_1^{-1} f_1 \doteq x \circ g_2^{-1} f_2$ implies $g_1^{-1} f_1 \doteq g_2^{-1} f_2$. Say (u, v) amalgamates (g_1, g_2). Let (r, s) amalgamate $(u \circ g_1, x)$. Notice that $x \circ g_1^{-1} f_1 = s^{-1}(r \circ u \circ f_1)$ and $x \circ g_2^{-1} f_2 = s^{-1}(r \circ v \circ f_2)$. From our assumption, we find that $r \circ u \circ f_1 = r \circ v \circ f_2$, and we are done by monicity of A.



As for stability, note that if h is $g^{-1}f$-stable, then $g \circ h$ is f-stable, and we have $(g^{-1}f)^{-1}h = f^{-1}(g \circ h)$; if $g^{-1}f$ is h-stable, then f is $g \circ h$-stable, and $h^{-1}(g^{-1}f) = (g \circ h)^{-1}f$. It becomes slightly more interesting to see what happens when $\pi^{-1}h$ is $g^{-1}f$-stable. Pick (r, s) amalgamating (π, g); then $g \circ (\pi^{-1}h) = s^{-1}(r \circ h)$. We find that $r \circ h$ is $(s \circ f)$-stable, and indeed, one can verify that $(g^{-1}f)^{-1}(\pi^{-1}h) = (s \circ f)^{-1} r \circ h$. □

Note that if A is countable, then so is $\bar{A}$.

Proposition 3.4. *If A, B are monic, Fraïssé categories and $A \overset{\sim}{\to} B$, then $\bar{A} \overset{\sim}{\to} \bar{B}$.*

Proof. Identify A with its image under a cofinal embedding. Let $f, g \in \mathrm{Arr}(A)$ have common range, and say f is g-stable in A. Suppose for $p_i \in \mathrm{Arr}(B)$ we have $p_1 \circ g = p_2 \circ g$. By cofinality, find x with $x \circ p_i \in \mathrm{Arr}(A)$. Then by stability we have $x \circ p_1 \circ f = x \circ p_2 \circ f$, and by monicity, we see that f is g-stable in B. □

4 Topological Groups

Let us start by reviewing some of the basic ideas about Fraïssé sequences in the countable case; for a more detailed exposition, see [K]. As an example to keep in mind, let $\mathcal{G}$ be the class of finite graphs, i.e. it is the category whose objects are finite graphs and whose arrows are embeddings. Here we use embedding in the model-theoretic sense: $f : H \to G$ is an embedding iff f is an injection with $E(u, v) \Leftrightarrow E(f(u), f(v))$.

We can consider the ordinal ω as a category whose objects are the finite ordinals and whose morphisms are inclusions. Given a category C, a *sequence* is a functor $u : \omega \to C$.

We will write u_n for $u(n)$, and for $n \subseteq m$, we write u_n^m for $u(i_n^m)$. In $\mathcal{G}$, this is a sequence of graphs G_n and embeddings $G_n^{n+1} : G_n \to G_{n+1}$ which, by identifying each G_n with its image in G_{n+1}, we may assume are inclusions. It is useful to think of this sequence as building a countably infinite graph $\cup_n G_n$.

For an object a, an arrow $a \to \vec{u}$ is an equivalence class of arrows $a \to u_i$; for $f \in \mathrm{Hom}(a, u_i)$, $g \in \mathrm{Hom}(a, u_j)$, we say $(f, i) \approx (g, j)$ if there is k with $u_i^k \circ f = u_j^k \circ g$. Equivalence classes will be denoted by $[f, i]$, or just $[f]$ if there is no confusion. In $\mathcal{G}$, an arrow $H \to \vec{G}$ is an embedding of H into $\cup_n G_n$. This also makes the role of $\approx$ clear; if $g : H \to G_n$ and $h = G_n^m \circ g$, then g and h give the same embedding of H into $\cup_n G_n$.

Given two sequences $\vec{u}, \vec{v}$, a *transformation* is a natural transformation $\varphi : \vec{u} \to \vec{v} \circ \pi$, where $\pi : \omega \to \omega$ is a functor. We will often abuse notation by identifying φ and π and writing $\varphi : \vec{u} \to \vec{v}$; this way we can simply write $\varphi_i \in \mathrm{Hom}(u_i, v_{\varphi(i)})$. Therefore we can think of φ as consisting of maps φ_i such that the following diagram commutes.

$$\begin{array}{ccccccc}
u_0 & \longrightarrow & u_1 & \longrightarrow & u_2 & \longrightarrow & \cdots \\
\Big\downarrow \varphi_0 & & \Big\downarrow \varphi_1 & & \Big\downarrow \varphi_2 & & \\
v_{\varphi(0)} & \longrightarrow & v_{\varphi(1)} & \longrightarrow & v_{\varphi(2)} & \longrightarrow & \cdots
\end{array}$$

We will most often consider the case $\vec{u} = \vec{v}$. If $\varphi, \psi : \vec{u} \to \vec{u}$ are transformations; we say $\varphi \approx \psi$ iff for all i we have $(\varphi_i, \varphi(i)) \approx (\psi_i, \psi(i))$. We will also denote these equivalence classes by $[\varphi]$. Composition of transformations is defined in the obvious way; if φ, ψ are transformations, then we set $(\psi \circ \varphi)_i = \psi_{\varphi(i)} \circ \varphi_i$. It is quick to check that this respects $\approx$. We can now define $\mathrm{Aut}(\vec{u})$ to be the group of invertible arrows. In $\mathcal{G}$, a transformation $\varphi : \vec{G} \to \vec{G}$ is just an embedding of $\cup_n G_n$ into $\cup_n G_n$, and $\mathrm{Aut}(\vec{G}) = \mathrm{Aut}(\cup_n G_n)$.

Now suppose C is a countable Fraïssé category; a *Fraïssé sequence* is a sequence $\vec{u}$ which satifsies the following two properties:

- Cofinality: for each $a \in \mathrm{Ob}(C)$, there is $i \in \mathbb{N}$ with $\mathrm{Hom}(a, u_i)$ nonempty.
- Extension Property (EP): for any arrows $f \in \mathrm{Hom}(a, u_i)$, $g \in \mathrm{Hom}(a, b)$, there is $j \in \mathbb{N}$ and $h \in \mathrm{Hom}(b, u_j)$ with $u_i^j \circ f = h \circ g$.

It is a fact that in countable Fraïssé categories, Fraïssé sequences exist and furthermore satisfy the *back and forth property*: let $\vec{u}, \vec{v}$ be Fraïssé sequences in a Fraïssé category C. Then for any arrows $f \in \mathrm{Hom}(a, u_i)$, $g \in \mathrm{Hom}(a, v_j)$, there is an isomorphism $\varphi : \vec{u} \to \vec{v}$ such that for some k, $v_{\varphi(i)}^k \circ \varphi_i \circ f = v_j^k \circ g$. In particular, any two Fraïssé sequences are isomorphic.

If $\vec{G}$ is a Fraïssé sequence in $\mathcal{G}$, then $\cup_n G_n \equiv \mathbf{G}$ is the *Rado graph*, often called the *random graph*. We can think of $\mathbf{G}$ as being built by taking a countably infinite number of vertices and flipping a fair coin for each pair to determine whether or not an edge is present. $\mathbf{G}$ is the unique countable graph up to isomorphism which embeds every finite graph and which satisfies the (EP): for all $n, m \in \mathbb{N}$ and any distinct vertices $x_1, ..., x_n, y_1, ..., y_m$, there is a vertex v distinct from the x_i, y_j with $E(v, x_i)$ and $\neg E(v, y_j)$ for $1 \leq i \leq n$, $1 \leq j \leq m$.

Our aim is to give $\mathrm{Aut}(\vec{u})$ the structure of a topological group. To do this, let us more closely investigate the action of transformations on arrows $a \to \vec{u}$. For $f \in \mathrm{Hom}(a, u_i)$, we write $\varphi(f) \equiv \varphi_i \circ f$.

Proposition 4.1. *Say $f \in \mathrm{Hom}(a, u_i)$, $g \in \mathrm{Hom}(a, u_j)$ with $(f, i) = (g, j)$, and suppose $\varphi \approx \psi$ are transformations. Then $(\varphi(f), \varphi(i)) \approx (\psi(g), \psi(j))$. Conversely, if φ, ψ are transformations such that $\varphi(a) \approx \psi(b)$ whenever $a \approx b$, we have $\varphi \approx \psi$.*

Proof. Say $u_i^k \circ f = u_j^k \circ g$ and $u^l_{\varphi(k)} \circ \varphi_k = u^l_{\psi(k)} \circ \psi_k$. We find that $\varphi_k \circ u_i^k \circ f \approx \psi_k \circ u_j^k \circ g$. As $\varphi_k \circ u_i^k = u^{\varphi(k)}_{\varphi(i)} \circ \varphi_i$ and $\psi_k \circ u_j^k = u^{\psi(k)}_{\psi(j)} \circ \psi_j$, we have $\varphi(f) \approx \psi(g)$ as desired. For the converse, apply the assumption to u_i^i. □

Corollary 4.2. *If $\varphi \in \mathrm{Aut}(\vec{u})$, then φ permutes the arrows $a \to \vec{u}$.*

We can now view $\mathrm{Aut}(\vec{u})$ as a subgroup of S_∞, the permutations of a countable set. S_∞ is a topological group with the topology of pointwise convergence; this allows us to give $\mathrm{Aut}(\vec{u})$ the subspace topology. More explicitly, a countable neighborhood basis of open subgroups at the identity is given by $\{\varphi \in \mathrm{Aut}(\vec{u}) : \varphi([u_i^i]) = [u_i^i]\}_{i \in \mathbb{N}}$.

Proposition 4.3. *If C is a countable Fraïssé category and $\vec{u}$ and $\vec{v}$ are two Fraïssé sequences in A, the resulting topological groups are homeomorphic.*

Proof. Let $\psi : \vec{u} \to \vec{v}$ be an isomorphism; then the map $\varphi \to \psi \circ \varphi \circ \psi^{-1}$ is an isomorphism $\mathrm{Aut}(\vec{u}) \to \mathrm{Aut}(\vec{v})$. Suppose $\varphi^{(n)} \to \varphi$ in the topology of $\mathrm{Aut}(\vec{u})$. This means that for each arrow $f : a \to \vec{u}$, there is $N \in \mathbb{N}$ such that for any $n > N$, we have $\varphi^{(n)}(f) = \varphi(f)$. But then we have $\psi \circ \varphi^{(n)} \circ \psi^{-1}(\psi(f)) = \psi \circ \varphi \circ \psi^{-1}(\psi(f))$; hence $\psi \circ \varphi^{(n)} \circ \psi^{-1} \to \psi \circ \varphi \circ \psi^{-1}$. □

Given a countable Fraïssé category C, we may now associate to it a unique topological group $G(C)$. Before continuing, it is worthwhile to look at the existing examples of topological groups arising from Fraïssé categories. The standard example of course is when the category C is a Fraïssé class such as our running example $\mathcal{G}$. Our requirements that a Fraïssé category be directed and have amalgamation then become the familiar *Joint Embedding Property* (JEP) and the *Amalgamation Property* (AP). The group $G(C)$ is just the automorphism group of the Fraïssé limit with the pointwise convergence topology (see [Ho]). In this way we see that each closed subgroup of S_∞ is a $G(C)$ for some countable category C (we will see shortly that $G(C)$ is always closed).

A more interesting example comes from the projective Fraïssé theory of Irwin and Solecki [IS]. Fix a countable language L; a *projection* of L-models $p : \mathbf{B} \to \mathbf{A}$ is a surjection $p : B \to A$ such that $p(f^B(x_1, ..., x_n)) = f^A(p(x_1), ..., p(x_n))$ for each function symbol $f \in L$ and $R^A(y_1, ..., y_n) \Leftrightarrow \exists(x_1, ..., x_n)[p(x_1, ..., x_n) = (y_1, ..., y_n) \wedge R^B(x_1, ..., x_n)]$ for each relation symbol $R \in L$. To turn a class of finite models into a Fraïssé category using projections for arrows, we reverse direction: each projection $p : \mathbf{B} \to \mathbf{A}$ corresponds to an arrow $p \in \mathrm{Hom}(a, b)$.

Given a Fraïssé sequence $\vec{\mathbf{A}}$ in a projective Fraïssé class $\mathcal{K}$, we can form the projective Fraïssé limit $\mathbf{K} \equiv \varprojlim(\mathbf{A}_n, A_n^m)$, a topological L-structure whose underlying set is the set-theoretic inverse limit of the sequence. For $\alpha_1, ..., \alpha_n \in \mathbf{K}$ and $f \in L$ a function symbol, we set $f^{\mathbf{K}}(\alpha_1, ..., \alpha_n) = \beta$ for the unique β with $f^{A_k}(\alpha_1(k), ..., \alpha_n(k)) = \beta(k)$ for each $k \in \mathbb{N}$. For $R \in L$ a relation symbol, we set $R^{\mathbf{K}}(\alpha_1, ..., \alpha_n)$ iff $R^{A_k}(\alpha_1(k), ..., \alpha_n(k))$ for each $k \in \mathbb{N}$. Each $f^{\mathbf{K}}$ is continuous, and each $R^{\mathbf{K}}$ is closed. Automorphisms of $\vec{\mathbf{A}}$ are exactly the structure-preserving homeomorphisms of $\mathbf{K}$. However, there is already a natural topology to put on the group of homeomorphisms, namely the compact-open topology. As $\mathbf{K}$ is a zero-dimensional Hausdorff space, a subbase for this topology is given by sets of the form $V(U_1, U_2) \equiv \{\varphi \in \text{Homeo}(\mathbf{K}) : \varphi(U_1) \subseteq U_2\}$ with each U_i clopen.

Proposition 4.4. *Let C be a countable Fraïssé category which is also a projective Fraïssé class $\mathcal{K}$. Let $\mathbf{K} = \varprojlim(\mathbf{A}_n, \pi_n)$ be the projective Fraïssé limit. Then $G(C) \cong \text{Homeo}(\mathbf{K})$ when $\text{Homeo}(\mathbf{K})$ is given the compact-open topology.*

Proof. A subbase for the topology on $G(C)$ is given by sets of the form $W(p_1, p_2) \equiv \{\varphi \in \text{Homeo}(\mathbf{K}) : p_1 \circ \varphi = p_2\}$, where the p_i are projections $\mathbf{K} \to \mathbf{A}$ for some $\mathbf{A} \in \mathcal{K}$. Then we see that $W(p_1, p_2)$ is exactly the intersection of open sets $V(p_1^{-1}(a), p_2^{-1}(a))$ for each $a \in \mathbf{A}$. Conversely, fix U_1, U_2 clopen and consider $V(U_1, U_2)$. As the U_i are clopen, we may find $\mathbf{A}_n \in \mathcal{K}$ such that there are $S_i \subseteq \mathbf{A}_n$ with $(A_n^\infty)^{-1}(S_i) = U_i$, where A_n^∞ is the natural projection $\mathbf{K} \to \mathbf{A}_n$. Now we see that $V(U_1, U_2)$ is the union of open sets of the form $W(A_n^\infty, p)$, where p is any projection $\mathbf{K} \to \mathbf{A}_n$ with $p(U_1) \subseteq S_2$. □

Proposition 4.5. *Suppose $A \xrightarrow{\sim} B$ are countable Fraïssé categories. Then $G(A) \cong G(B)$.*

Proof. We will show instead on A that if $\varphi^{(n)}$ converges to φ on all arrows of the form $u_i \to \vec{u}$, then $\varphi^{(n)}$ converges to φ on all arrows. To see this, let $[f] : a \to \vec{u}$ be an arrow; pick a representative $f : a \to u_i$. By assumption, $\varphi^{(n)}$ is eventually constant on the arrow $[u_i^i]$. It follows that $\varphi^{(n)}$ is eventually constant on $[f]$. As any Fraïssé sequence for A is also a Fraïssé sequence for B, we are done. □

Proposition 4.6. *For C a countable Fraïssé category, $G(C)$ is a closed subgroup of S_∞.*

Proof. Suppose $\varphi^{(n)}$ converges pointwise to a permutation of all arrows $a \to \vec{u}$. Write $\psi^{(n)} \equiv (\varphi^{(n)})^{-1}$. Note that $\psi^{(n)}$ also converges pointwise. Pick $\varphi(1) \in \mathbb{N}$ and $\varphi_1 : u_1 \to u_{\varphi(1)}$ such that $[\varphi_1] = \varphi^{(n)}([u_1^1])$ for sufficiently large n. Inductively Pick $\varphi(k) > \varphi(k-1)$ and an arrow $\varphi_k : u_k \to u_{\varphi(k)}$ with $[\varphi_k] = \varphi^{(n)}([u_k^k])$ for large n. Note that $u_{\varphi(j)}^{\varphi(k)} \circ \varphi_j = \varphi_k \circ u_j^k$ for $j < k$. We build $\psi \equiv \varphi^{-1}$ in the exact same way. □

Proposition 4.7. *For C a countable Fraïssé category, $G(C) \cong G(\bar{C})$.*

Proof. Naturally, this proof will be done in two steps. We first show that $G(C) \cong G(C/M)$; then assuming C is monic, we show that $G(C) \cong G(\bar{C})$. First, note that the map $[f] \to [Mf]$ is a bijection. For suppose $[Mf] = [Mg]$; then there is k with $Mu_i^k \circ Mf = Mu_j^k \circ Mg$. Thus there is y with $y \circ u_i^k \circ f = y \circ u_j^k \circ g$. Using the extention property, we are done.

Much in the same way, we can show that the map $[\varphi] \to [M\varphi]$ is an injection; the action of $[M\varphi]$ on the arrows $[Mf]$ is identical to the action of $[\varphi]$ on arrows $[f]$. It only remains to show that this map is surjective. So assume ψ is an automorphism of $\vec{u}$ in C/M. We will build an automorphism φ of $\vec{u}$ in C such that $M\varphi \approx \psi$. Choose $y_i \in \psi_i$,

$i \in \omega$. Then for each $i < j$ there is $x(i,j)$ with $x(i,j) \circ u_{\psi(i)}^{\psi(j)} \circ y_i = x(i,j) \circ y_j \circ u_i^j$. By the extention property, we can find $k(i,j) \in \omega$ and $w(i,j)$ with $w(i,j) \circ x(i,j) = u_{\psi(j)}^{k(i,j)}$. Define $\varphi_j = u_{\psi(j)}^{k(j)} \circ y_j$, where $k(0) = \psi(0)$, and for $j > 0$, $k(j) = \max(k(i,j)_{i<j}, (k(i)+1)_{i<j})$.

Now assume C is monic. First let us show that passing to $\bar{C}$ introduces no new arrows $a \to \vec{u}$. For suppose we have $g^{-1}f : a \to u_i$. By the extension property, there is x with $x \circ g = u_i^k$. We see that $g^{-1}f \approx x \circ f$. We will be done once we also show that there are no new isomorphisms. Let ψ be an isomorphism; inductively choose φ_i with $\varphi_i \in \mathrm{Arr}(C)$, $\varphi_i \approx \psi_i$, and $i < j \Rightarrow \varphi(i) < \varphi(j)$. Hence $\psi \approx \varphi$. □

5 Topological Fraïssé Classes

The goal of this section is to concretely describe all monic stable Fraïssé categories. Along the way, we will prove:

Theorem 5.1. *Let A, B be countable Fraïssé categories with $G(A) \cong G(B)$. Then we have $\bar{A} \sim \bar{B}$.*

Fix G a closed subgroup of S_∞, and let β be a countable (perhaps finite) collection of open subgroups whose conjugates form a neighborhood basis at the identity. We allow β to contain duplicates; for short call such a β a *G-basis*. We define the category $C(\beta)$ as follows. We set $\mathrm{Ob}(C(\beta)) = \beta$. An arrow $U \to V$ is a pair (V, gU), with $V \subseteq gUg^{-1}$. We set $(W, hV) \circ (V, gU) = (W, hgU)$; the requirement that $V \subseteq gUg^{-1}$ ensures that this is well defined.

Proposition 5.2. *$C(\beta)$ is a countable, monic, stable Fraïssé category.*

Proof. If $U, V \in \beta$, then there is $W \in \beta$ and $g \in G$ with $g^{-1}Wg \subseteq U \cap V$. Hence $C(\beta)$ is directed. If (V_1, g_1U) and (V_2, g_2U) are arrows, then pick $W \in \beta$ and $g \in G$ with $g^{-1}Wg \subseteq g_1^{-1}V_1g_1 \cap g_2^{-1}V_2g_2$; we see that $(W, gg_1^{-1}V_1) \circ (V_1, g_1U) = (W, gg_2^{-1}V_2) \circ (V_2, g_2U)$. Hence $C(\beta)$ has amalgamation. As for each $U \in \beta$ there are only countably many left cosets gU, it follows that $C(\beta)$ is countable.

Suppose $(W, hV) \circ (V, g_1U) = (W, hV) \circ (V, g_2U)$. Then $hg_1U = hg_2U$, and $g_1U = g_2U$. Hence $C(\beta)$ is monic. Lastly, suppose (V, g_1U_1) is (V, g_2U_2)-stable. It follows that if $h_1g_2U_2 = h_2g_2U_2$, then $h_1g_1U_1 = h_2g_1U_1$. In particular, we have $h_2^{-1}h_1 \in g_1U_1g_1^{-1}$ whenever $h_2^{-1}h_1 \in g_2U_2g_2^{-1}$, hence $g_2U_2g_2^{-1} \subseteq g_1U_1g_1^{-1}$. We see that $(V, g_1U_1) = (V, g_2U_2) \circ (U_2, g_2^{-1}g_1U_1)$, and $C(\beta)$ is stable. □

We will call a category formed in the above manner a *topological Fraïssé class.*

Proposition 5.3. *Every countable, monic, stable, Fraïssé category is isomorphic to a topological Fraïssé class.*

Proof. Let C be such a category, and let $\vec{u}$ be a Fraïssé sequence. Write $G \equiv \mathrm{Aut}(\vec{u})$. For $a \in \mathrm{Ob}(C) \setminus \{u_i : i \in \omega\}$, pick an arrow $[f_a] : a \to \vec{u}$ arbitrarily. For $u \in \{u_i : i \in \omega\}$, find the least i such that $u = u_i$, and set $[f_u] = [u_i^i]$. Now for each $a \in \mathrm{Ob}(C)$, let

$U_a = \{\varphi \in G : \varphi([f_a]) = [f_a]$, and form the G-basis $\beta = \{U_a : a \in \mathrm{Ob}(C)\}$, allowing duplicates (so if $U_a = U_b$ for $a \neq b$, we consider U_a and U_b to be distinct elements of β).

Form the functor $\psi : C \to C(\beta)$ as follows. Set $\psi(a) = U_a$. For $x \in \mathrm{Hom}(a, b)$, set $\psi(x) = (U_b, gU_a)$, where $gU_a = \{\varphi \in G : \varphi([f_a]) = [f_b \circ x]\}$. Note that if $\varphi \in U_b$, then $\varphi([f_b \circ x]) = [f_b \circ x]$, so we have $U_b \subseteq gU_a g^{-1}$ as required. If $\psi(y) = (U_c, hU_b)$, then $\psi(y) \circ \psi(x) = (U_c, hgU_a)$; as $hgU_a = \{\varphi \in G : \varphi([f_a]) = [f_c \circ y \circ x]\}$, we have $\psi(y) \circ \psi(x) = \psi(y \circ x)$.

As ψ is certainly a bijection on objects, it only remains to show that it is a bijection on arrows. Suppose $\psi(w) = \psi(x) = (U_b, gU_a)$. Then in particular we have $g([f_a]) = [f_b \circ x] = [f_b \circ w]$. From monicity, it follows that $w = x$. Now say (U_b, gU_a) is an arrow. Set $[x] = g([f_a])$, with $x \in \mathrm{Hom}(a, u_j)$. If $f_b \in \mathrm{Hom}(b, u_i)$, we may assume $i < j$. I claim that x is $(u_i^j \circ f_b)$-stable. Indeed, suppose $p_1 \circ u_i^j \circ f_b = p_2 \circ u_i^j \circ f_b$; we may suppose $p_\ell \in \mathrm{Hom}(u_j, u_k)$ for some k. Pick $h \in G$ with $h([f_b]) = [p_\ell \circ u_i^j \circ f_b]$. We see that there must be $\varphi^{(1)}, \varphi^{(2)} \in hU_b$ with $\varphi_j^{(\ell)} \approx p_\ell$. But since (U_b, gU_a) is an arrow, we have $U_b \subseteq gU_a g^{-1}$; it follows that $\varphi^{(1)}([x]) = \varphi^{(2)}([x])$. Hence $[p_1 \circ x] = [p_2 \circ x]$, so by monicity $p_1 \circ x = p_2 \circ x$. By stability, write $x = u_i^j \circ f_b \circ y$. Then $\psi(y) = (U_b, gU_a)$. □

Proof of Theorem 5.1. Let $\bar{A} \cong C(\beta_1)$ and $\bar{B} \cong C(\beta_2)$. Let $C = C(\beta_1 \cup \beta_2)$. □

Corollary 5.4. *If G is a closed subgroup of S_∞ and β is a G-basis, then $G(C(\beta)) \cong G$.*

Proof. Let C be a countable Fraïssé category with $G(C) = G$. Let γ be a G-basis with $\bar{C} \cong C(\gamma)$. Then by considering the category $C(\beta \cup \gamma)$, we see that $C \sim C(\beta)$. □

6 Monoids

A *monoid* is a category with a single object. Alternatively and equivalently, it is a set with a binary, associative operation and an identity with respect to that operation. If M is a monoid and $I \subseteq M$, we say I is a left (right) ideal if I is closed under left (right) multiplication by elements of M. We say I is a *principal* left ideal if $I = Ms$ for some $s \in M$. We can now rephrase the Fraïssé condition in a way that will be convenient for discussing monoids; a monoid is Fraïssé iff every pair of nonempty principal left ideals has nonempty intersection. In the literature, this is sometimes called Ore's condition or right reversibility [CP]. We will also say *left cancellative* in place of monic, with *right cancellative* defined analogously; a monoid is *cancellative* if it is both left and right cancellative.

As applied to monoids, the previous part of the paper seems like a curious black box: input a countable Fraïssé monoid, and receive as output a possibly quite complicated topological group. The goal of this section is to characterize the Fraïssé group of a monoid. As a warmup, let us see that quite often, the Fraïssé group is relatively uninteresting.

Proposition 6.1. *Suppose a countable Fraïssé monoid M embeds into a group G and that M generates G. Then $G(M) = G$ with the discrete topology.*

In particular, this is the case iff M is cancellative. This is certainly necessary. If M is cancellative, then we see that for each $g \in M$, 1 is g-stable, and $\overline{M}$ is the desired group.

Proof. By passing to $\overline{M}$, we may assume $M = G$. Fix a Fraïssé sequence $\vec{u}$. First observe that if we restrict an automorphism φ to have $\varphi(i) = i$ for each $i \in \omega$, then by setting $\varphi_0 = g$ for some $g \in G$, then φ_i is uniquely determined. This follows from the constraint $\varphi_i \cdot u_0^i = u_0^i \cdot \varphi_0$ and the fact that everything is invertible. Now suppose φ is any automorphism. We want to build an automorphism ψ satisfying the restriction $\psi(i) = i$ with $\varphi \approx \psi$. There is only one possible choice for ψ_0, namely $\psi_0 = (u_0^{\varphi(0)})^{-1}\varphi_0$. This uniquely determines each ψ_k, and we have $\varphi_k = u_k^{\varphi(k)}\psi_k$, showing that $\varphi \approx \psi$.

Now for φ, ψ with $\varphi(i) = \psi(i) = i$, we have $\psi_0 \circ \varphi_0 = (\psi \circ \varphi)_0$, hence algebraically $G(M) \cong G$. To see that $G(M)$ is discrete, suppose an automorphism φ fixes the arrow $[u_0^0]$. Then $\varphi_0 = 1_G$; hence $\{1_G\}$ is open. □

As $\overline{M}$ is always left cancellative, we should look at monoids which are not right cancellative to find examples where $G(M)$ is quite complicated. Let's consider the monoid T of order-preserving, almost onto injections $\mathbb{N} \to \mathbb{N}$. T is not right cancellative; in fact, $\overline{T} = T$, but this will not be important in the sequel. We will use $\mathbb{N}$ to refer to the unique object of the category T.

Proposition 6.2. *T is a countable Fraïssé monoid.*

Proof. That T is countable is immediate. To show that T is Fraïssé, we want to show that for all $s, t \in T$, there is (u, v) amalgamating (s, t). Define $f : \mathbb{N} \to \mathbb{N}$ as follows (f will become $u \circ s = v \circ t$).

$$f(1) = \max(s(1), t(1))$$
$$f(n+1) = f(n) + \max(s(n+1) - s(n), t(n+1) - t(n))$$

Now define u, v as follows.

$$u(n) = \begin{cases} n & \text{if } n < s(1) \\ f(k) + n - s(k) & \text{if } s(k) \le n < s(k+1) \end{cases}$$
$$v(n) = \begin{cases} n & \text{if } n < t(1) \\ f(k) + n - t(k) & \text{if } t(k) \le n < t(k+1) \end{cases}$$

We see that $u \circ s = v \circ t = f$. Furthermore, u and v are both increasing injections. Since s and t are almost onto, f is almost onto, from which it follows that u and v are almost onto. □

The proof of proposition 6.2 shows us something more general; define

$$f_n(m) = \begin{cases} nm & \text{if } n \le m \\ f_n(n) + m - n & \text{if } n > m. \end{cases}$$

Then we see that for any $t \in T$, there is $n \in \mathbb{N}$ and $v \in T$ such that $f_n = v \circ t$. Even more generally, if $f_n \in Tt$ and $g : \mathbb{N} \to \mathbb{N}$ is any almost onto increasing injection such that $g(1) \ge n$ and $g(k) - g(k-1) \ge n$ for $k \le n$, then we likewise have $g \in Tt$. Pursuing this idea will allow us to concretely construct a Fraïssé sequence in T.

Enumerate $\mathbb{Q} \setminus \mathbb{N} = \{q_1, q_2, ...\}$. Set $N_0 = \mathbb{N}$, $N_k = \mathbb{N} \cup \{q_1, ..., q_k\}$. Henceforth we will refer to the copy of $\mathbb{N} \in \mathbb{Q}$ by N_0, using $\mathbb{N}$ to refer to the single object of the category T. We will view the inclusion $N_k \hookrightarrow N_{k+1}$ as representing the embedding $u_k^{k+1} : \langle N_{k+1}, < \rangle \to \langle N_{k+1}, < \rangle$ with $\mathrm{Ran}(u_k^{k+1}) = N_k$. In particular, we can view u_k^{k+1} as an element of T by putting N_{k+1} in bijection with $\mathbb{N}$, and u_k^ℓ gives the embedding $\langle N_\ell, < \rangle \to \langle N_\ell, < \rangle$ with $\mathrm{Ran}(u_k^\ell) = N_k$. From our above observation, we see that $\vec{u}$ is a Fraïssé sequence. Let $H \leq \mathrm{Aut}(\langle \mathbb{Q}, < \rangle)$ be the subgroup consisting of those automorphisms which almost fix N_0 setwise. Endow H with the topology τ, which is generated by the pointwise convergence topology and the additional open sets $U_X = \{h \in H : h(N_0) = X\}$.

Theorem 6.3. $G(T) \cong (H, \tau)$

Proof. Much in the way that we can identify Fraïssé sequences of models and model-theoretic embeddings with a structure (e.g. the random graph), we can identify $\vec{u}$ with $\langle \mathbb{Q}, < \rangle = \bigcup_n N_n$. In this case, an arrow $f : \mathbb{N} \to \vec{u}$ is an embedding f of $\mathbb{N}$ into $\langle \mathbb{Q}, < \rangle$ with $\mathrm{Ran}(f) \triangle N_0$ finite. As automorphisms act on arrows, we have $\mathrm{Aut}(\vec{u}) \subset H$. Conversely, given $h \in H$, we see that for each i, $h(N_i) \subseteq N_j$ for some j. Set $\varphi(i) = j$, and pick $\varphi_i \in T$ such that $\varphi_i : \langle N_j, < \rangle \to \langle N_j, < \rangle$ has range $h(N_i)$. Noting that the topology is given by pointwise convergence of arrows $\mathbb{N} \to \vec{u}$, we see that τ is the correct topology. □

Using this structural approach, we can describe $G(M)$ for every countable Fraïssé monoid M. Assume M is left cancellative. Let $\mathcal{R}_M$ denote the *right action structure* of M; i.e. $\mathcal{R}_M = \langle M, \{R_t : t \in M\} \rangle$, where $R_t(s) = st$. Now the maps $L_t : \mathcal{R}_M \to \mathcal{R}_M$ with $L_t(s) = ts$ are model-theoretic embeddings. In particular, $L_s \circ L_t = L_{st}$. Furthermore, if $\varphi : \mathcal{R}_M \to \mathcal{R}_M$ is any embedding (even any homomorphism), then letting $t = \varphi(1)$, we have $\varphi = L_t$. Hence M is the monoid of endomorphisms of $\mathcal{R}_M$.

Let $\vec{u}$ be a Fraïssé sequence in M. Form the directed system $(\mathcal{R}_i, u_j^i)$, where each R_i is R_M and $u_j^i \in M$ is viewed as an embedding $R_M \to R_M$. Let X be the direct limit.

If Y is a discrete space on which M acts on the right, we say that Y is directed if for all $x, y \in Y$, there is $z \in Y$ and $s, t \in M$ with $zs = zt$.

Theorem 6.4. *The action (X, M) is universal for countable directed actions and homogeneous: for each $x, y \in X$, there is an automorphism φ of (X, M) with $\varphi(x) = y$. Furthermore, (X, M) is the unique such action up to isomorphism.*

Proof. Seeing as we may view any countable directed action as some direct limit of a countable directed system, universality follows from the fact that the Fraïssé sequence is universal for countable sequences from M (see [K]). Homogeneity follows from the fact that we may associate to x and y the actions xM and yM, each of which is isomorphic to $\mathcal{R}_M$. Hence each of these actions corresponds to an arrow $\mathcal{R}_M \to \vec{u}$, and hence an automorphism sending x to y exists by the back and forth property. Uniqueness follows from uniqueness of the Fraïssé limit. □

Corollary 6.5. $G(M) \cong \mathrm{Aut}(X, M)$ *with the pointwise convergence topology.*

As an application, we will show that for M a countable Fraïssé monoid, $G = G(M)$ does not have metrizable universal minimal flow, exhibiting a large new class of groups with this property (see [KPT] for a discussion). Form the space $(\beta M)^X$, where βM is the Stone-Čech compactification of M. Recall that βM is extremely disconnected; therefore it suffices to find a minimal flow of G which projects onto an infinite subset of βM.

G acts by homeomorphisms on $(\beta M)^X$ via shift; i.e. $g \cdot \alpha(x) = \alpha(g^{-1}(x))$. We need to take some care here as there are two relevant topologies on G; the pointwise convergence topology which arises from G acting by automorphisms on (X, M), and the subspace topology inherited from Homeo$((\beta M)^X)$ with the compact-open topology. While the continuity of G-action on $(\beta M)^X$ would be immediate in the compact-open topology, we are concerned with the pointwise convergence topology. A base of open sets of $(\beta M)^X$ is given by sets of the form $((A_1, x_1), ..., (A_n, x_n))$, where $x_i \in X$, $A_i \subseteq M$, and this denotes those $\alpha \in (\beta M)^X$ for which $\alpha(x_i) \in \overline{A_i}$ for each $1 \leq i \leq n$. So suppose $g \cdot \alpha \in ((A_1, x_1), ..., (A_n, x_n))$. As (X, M) is directed, find y such that there are $t_i \in M$ with $yt_i = x_i$. Let $U(g^{-1}(y), y)$ denote the open subset of G consisting of those $h \in G$ for which $hg^{-1}(y) = y$. Then we see that for any $(h, \gamma) \in U(g^{-1}(y), y) \times ((A_1, g^{-1}(x_1)), ..., (A_n, g^{-1}(x_n)))$, we have $h \cdot \gamma \in ((A_1, x_1), ..., (A_n, x_n))$. Therefore G-action is continuous, and $(\beta M)^X$ is a G-flow.

The next two lemmas together will prove the result. The first shows that the result holds modulo a condition on the algebraic structure of βM. The second states that we may restrict our attention to those M where this condition holds.

Lemma 6.6. *Let $t \in M$ and suppose $I \subset \beta M$ is infinite and minimal with respect to I being closed and $It \subseteq I$. Then G has nonmetrizable universal minimal flow.*

Proof. First observe that as I is minimal, we have $It = I$. In particular, $R_t^{-1}(p) \neq \emptyset$ for any $p \in I$. By the assumption that I is infinite, we must have $t^k \neq t^\ell$ for $k \neq \ell$. Find $\alpha \in (\beta M)^X$ with $\alpha(x) \in I$, $\alpha(xt) = \alpha(x)t$ for all $x \in X$. Let Y be a minimal flow of $\overline{G \cdot \alpha}$. Then for any $x \in X$, the projection of Y onto the x-coordinate is I. As I is infinite and βM is extremely disconnected, I is not metrizable, hence Y is not metrizable. □

Lemma 6.7. *Let $t \in M$ be such that $t^k \neq t^\ell$ for $k \neq \ell$. Then there is an I as in the above lemma.*

In particular, if M is left cancellative and there is no such t, then M is a group, $G = M$, and all infinite countable discrete groups are known to have nonmetrizable universal minimal flow.

Proof. Identifying βM with the space of ultrafilters on M, we can identify the closed subsets of βM with filters on M. In particular, if $\mathcal{F}$ is a filter on T, then $C_\mathcal{F} = \{\mathcal{U} : \mathcal{F} \subseteq \mathcal{U}\}$, i.e. those ultrafilters extending $\mathcal{F}$. Set $T = \{1, t, t^2, ...\}$, and consider $I \subseteq \beta T \subseteq \beta M$ minimal with respect to being closed and with $It \subseteq I$. Let $I = C_\mathcal{F}$. Since $It = I$ by minimality, we must have $R_t^{-1}(B) \in \mathcal{F}$ for each $B \in \mathcal{F}$. Let $\mathcal{F}t$ denote the filter with base $\{Bt : B \in \mathcal{F}\}$. Now we have $C_\mathcal{F}t \subseteq C_{\mathcal{F}t} \subseteq C_\mathcal{F}$, hence $\mathcal{F}t = \mathcal{F}$. Now it only remains to observe that this implies $\{t^m, t^{m+k}, t^{m+2k}, ...\} \notin \mathcal{F}$, which in turn implies that there are at least k distinct ultrafilters extending $\mathcal{F}$ for each $k \in \mathbb{N}$, showing that I is infinite. □

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